

# Two-Sided Power Random Variables

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## Abstract

We study a well-known problem concerning a random variable  $Z$  uniformly distributed between two independent random variables. Two different extensions, conditionally directed power distribution and conditionally undirected power distribution, have been introduced for this problem. For the second method, two-sided power random variables have been defined.

## 1 Introduction

Van Assche (1987) introduced the notion of a random variable  $Z$  uniformly distributed between two independent random variables  $X_1$  and  $X_2$ , which arose in studying the distribution of products of random  $2 \times 2$  matrices for stochastic search of global maxima. By letting  $X_1$  and  $X_2$  to have identical distributions, he derived that: (i) for  $X_1$  and  $X_2$  on  $[-1, 1]$ ,  $Z$  is uniform on  $[-1, 1]$  if and only if  $X_1$  and  $X_2$  have an Arcsin distribution; and (ii)  $Z$  possesses the same distribution as  $X_1$  and  $X_2$  if and only if  $X_1$  and  $X_2$  are degenerated or have a Cauchy distribution. Soltani and Homei (2009) following Johnson and Kotz (1990) extended Van Assche's results. They put  $X_1, \dots, X_n$  to be independent, and considered

$$S_n = R_1X_1 + R_2X_2 + \dots + R_{n-1}X_{n-1} + R_nX_n, \quad n \geq 2,$$

where random proportions are  $R_i = U_{(i)} - U_{(i-1)}$ ,  $i = 1, \dots, n-1$  and  $R_n = 1 - \sum_{i=1}^{n-1} R_i$ ,  $U_{(1)}, \dots, U_{(n-1)}$  are order statistics from a uniform distribution on  $[0, 1]$ , and  $U_{(0)} = 0$ . These random proportions are uniformly distributed over the unit simplex. They employed Stieltjes transform and derived that: (i)  $S_n$  possesses the same distribution as  $X_1, \dots, X_n$  if and only if  $X_1, \dots, X_n$  are degenerated or have a Cauchy distribution; and (ii) Van Assche's (1987) result for Arcsin holds for  $Z$  only.

In this paper, we introduce two families of distributions, suggested by an anonymous referee of the article, to whom the author expresses his deepest gratitude. We say that  $Z_1$  is a random variable between two independent random variables with power distribution, if the conditionally

distribution of  $Z_1$  given at  $X_1 = x_1, X_2 = x_2$  is

$$F_{Z_1|x_1, x_2}(z) = \begin{cases} 1 & z \geq \max(x_1, x_2), \\ \left(\frac{z-x_1}{x_2-x_1}\right)^n & x_1 < z < x_2, \\ 1 - \left(\frac{z-x_1}{x_2-x_1}\right)^n & x_2 < z < x_1, \\ 0 & z \leq \min(x_1, x_2). \end{cases} \quad (1.1)$$

The distribution  $F_{Z_1|x_1, x_2}$  will be said to follow a conditionally directed power distribution, when  $n$  is an integer. For  $n = 1$ , the distribution given by (1.1) simplifies to the distribution  $Z$  that was introduced by Van Assche (1987). For  $n = 2$ , we call  $Z_1$  directed triangular random variable. For further generalizing Van Assche results, we introduce a seemingly more natural conditionally power distribution. We call  $Z_2$  two-sided power (TSP) random variable, if the conditionally distribution of  $Z_2$  given at  $X_1 = x_1, X_2 = x_2$  is

$$F_{Z_2|x_1, x_2}(z) = \begin{cases} 1 & z \geq y_2, \\ \left(\frac{z-y_1}{y_2-y_1}\right)^n & y_1 < z < y_2, \\ 0 & z \leq y_1. \end{cases} \quad (1.2)$$

The distribution  $F_{Z_2|x_1, x_2}$  will be said to follow a conditionally undirected power distribution, when  $y_1 = \min(x_1, x_2), y_2 = \max(x_1, x_2)$  and  $n$  is an integer. For  $n = 2$ , we call  $Z_2$  undirected triangular random variable.

Again, for  $n = 1$ , the distribution given by (1.1) simplifies to the distribution  $Z$  that was introduced by Van Assche (1987). The main aim of this article is providing a couple of generalizations to the results of Van Assche (1987) for some other values of  $n$  (other than  $n = 1$ ). This article is organized as follows. We introduce preliminaries and previous works in section 2. In section 3, we give some characterizations for distribution  $Z_1$  given in (1.1), when  $n = 2$ . In section 4, we find distribution of  $Z_2$  given in (1.2) by direct method, and give some examples of such distributions.

## 2 Preliminaries and previous works

In this section, we first review some results of Van Assche (1987) and then modify them a little bit to fit in our framework, to be introduced in the forthcoming sections.

Using the Heaviside function ( $U(x) = 0, x < 0, = 1, x \geq 0$ ) we conclude that for any given distinct values  $x_1$  and  $x_2$ , the conditional distribution  $F_{Z_1|x_1, x_2}(z)$  in (1.1) is

$$F_{Z_1|x_1, x_2}(z) = \left(\frac{z - x_1}{x_2 - x_1}\right)^n U(z - x_1) - \sum_{i=1}^n \binom{n}{i} \left(\frac{z - x_2}{x_2 - x_1}\right)^i U(z - x_2). \quad (2.1)$$

**Lemma 2.1.** For distinct reals  $x_1, x_2, z$  and integer  $n$ , we have

$$\frac{-1}{(z - x_1)(x_2 - x_1)^n} + \frac{(-1)^n}{(n - 1)!} \frac{d^{n-1}}{dx_2^{n-1}} \left( \frac{1}{z - x_2} \cdot \frac{1}{(x_1 - x_2)} \right) = \frac{1}{(x_1 - z)(x_2 - z)^n}.$$

**Proof.** It easily follows from the Leibniz formula. □

Another tool for proving our main theorem is the following formula taken from the Schwartz distribution theory, namely,

$$\int_{-\infty}^{\infty} \varphi(x) \Lambda^{[n]}(dx) = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} \frac{d^n}{dx^n} \varphi(x) \Lambda(dx), \quad (2.2)$$

where  $\Lambda$  is a distribution function and  $\Lambda^{[n]}$  is the  $n$ -th distributional derivative of  $\Lambda$ .

The conditional distribution  $F_{Z_1|x_1, x_2}(z)$  given by (1.1) leads us to a linear functional on complex-valued functions  $f : \mathbb{R} \rightarrow \mathbb{C}$ , defined on the set of real numbers  $\mathbb{R}$ :

$$F_{Z_1|x_1, x_2}(f) = \frac{f(x_1)}{(x_2 - x_1)^n} - \sum_{i=1}^n \frac{1}{(n - i)!(x_2 - x_1)^i} \frac{d^{n-i}}{dx^{n-i}} f(x_2).$$

It easily follows that

$$F_{Z_1|x_1, x_2}(af + bg) = aF_{Z_1|x_1, x_2}(f) + bF_{Z_1|x_1, x_2}(g), \quad (2.3)$$

for any complex-valued functions  $f, g$  and complex constants  $a, b$ . We note that  $F_{Z_1|x_1, x_2}(z) = F_{Z_1|x_1, x_2}(f_z)$ , whenever  $f_z(x) = (z - x)^n U(z - x)$  and

$$F_{Z_1|x_1, x_2}(f_z) = \frac{f_z(x_1)}{(x_2 - x_1)^n} - \sum_{i=1}^n \frac{1}{(n - i)!(x_2 - x_1)^i} \frac{d^{n-i}}{dx^{n-i}} f_z(x_2).$$

Also we note that  $U(z - x) = \frac{(-1)^n}{(n)!} \frac{d^n}{dx^n} f_z(x)$ . Thus

$$P(Z_1 \leq z) = \int_{\mathbb{R}} U(z - x) dF_{Z_1}(x) = \int_{\mathbb{R}^2} F_{Z_1|x_1, x_2}(z) \prod_{i=1}^2 F_{X_i}(dx_i),$$

can be viewed as:

$$\int_{\mathbb{R}} \frac{(-1)^n}{(n)!} \frac{d^n}{dx^n} f_z(x) dF_{Z_1}(x) = \int_{\mathbb{R}^2} F_{Z_1|x_1, x_2}(f_z) \prod_{i=1}^2 F_{X_i}(dx_i). \quad (2.4)$$

Therefore by using (2.3) along with (2.4) and a standard argument in the integration theory, we obtain that

$$\int_{\mathbb{R}} \frac{(-1)^n}{(n)!} \frac{d^n}{dx^n} f(x) dF_{Z_1}(x) = \int_{\mathbb{R}^2} F_{Z_1|x_1, x_2}(f) \prod_{i=1}^2 F_{X_i}(dx_i), \quad (2.5)$$

for any infinitely differentiable functions  $f$  for which the corresponding integrals are finite. Now (2.5) together with (2.2) lead us to

$$\int_{\mathbb{R}} f(x) dF_{Z_1}^{(n)}(x) = \int_{\mathbb{R}^2} F_{Z_1|x_1, x_2}(f) \prod_{i=1}^2 F_{X_i}(dx_i), \quad (2.6)$$

for the above mentioned functions  $f$ , where  $F_{Z_1}^{(n)}$  is the  $(n)$ -th distributional derivative of the distribution of  $Z_1$ .

Let us denote the Stieltjes transform of a distribution  $H$  by

$$\mathcal{S}(H, z) = \int_{\mathbb{R}} \frac{1}{z - x} H(dx),$$

for every  $z$  in the set of complex numbers  $\mathbb{C}$  which does not belong to the support of  $H$ , i.e.,  $z \in \mathbb{C} \cap (\text{supp} H)^c$ . For more on the Stieltjes transform, see Zayed (1996).

The following lemma indicates how the Stieltjes transform of  $Z_1$ , and  $X_1, X_2$  are related.

**Lemma 2.2.** Let  $Z_1$  be a random variables that satisfies (1.1). Suppose that the random variables  $X_1$  and  $X_2$  are independent and continuous with distribution functions  $F_{X_1}$  and  $F_{X_2}$ , respectively. Then

$$\frac{1}{n} \mathcal{S}^{(n)}(F_{Z_1}, z) = -\mathcal{S}(F_{X_1}, z) \mathcal{S}^{(n-1)}(F_{X_2}, z), \quad z \in \mathbb{C} \bigcap_{i=1}^2 (\text{supp} F_{X_i})^c.$$

**Proof.** It follows from (2.6) that

$$\mathcal{S}(F_{Z_1}^{(n)}, z) = \int_{\mathbb{R}^2} F_{Z_1|x_1, x_2}(g_z) \prod_{i=1}^2 F_{X_i}(dx_i),$$

and

$$\frac{1}{n!} \frac{d^n}{dz^n} \mathcal{S}(F_{Z_1}, z) = \int_{\mathbb{R}^2} F_{Z_1|x_1, x_2}(g_z) \prod_{i=1}^2 F_{X_i}(dx_i),$$

for  $g_z(x) = \frac{1}{z-x}$ . Now, it follows that

$$F_{Z_1|x_1,x_2}(g_z) = \frac{\frac{1}{z-x_1}}{(x_2-x_1)^n} - \sum_{i=1}^n \frac{1}{(n-i)!(x_2-x_1)^i} \frac{d^{n-i}}{dz^{n-i}} \frac{1}{z-x_2},$$

and by using Lemma 2.1, we have

$$F_{Z_1|x_1,x_2}(g_z) = \frac{(-1)^n}{(z-x_1)(z-x_2)^n}.$$

Therefore,

$$\frac{1}{n!} \frac{d^n}{dz^n} \mathcal{S}(F_{Z_1}, z) = \int_{\mathbb{R}^2} \frac{(-1)^n}{(z-x_1)(z-x_2)^n} \prod_{i=1}^2 F_{X_i}(dx_i),$$

and

$$\frac{1}{n} \mathcal{S}^{(n)}(F_{Z_1}, z) = -\mathcal{S}(F_{X_1}, z) \mathcal{S}^{(n-1)}(F_{X_2}, z), \quad z \in \mathbb{C} \bigcap_{i=1}^2 (\text{supp } F_{X_i})^c. \quad (2.7)$$

This finishes the proof.  $\square$

Note that Van Assche's lemma is the case of  $n = 1$ :

$$-\mathcal{S}'(F_{Z_1}, z) = \mathcal{S}(F_{X_1}, z) \mathcal{S}(F_{X_2}, z).$$

We also note that the Stieltjes transform of Cauchy distribution, i.e.,  $\mathcal{S}(F, z) = \frac{1}{z+c}$ , satisfies (2.7).

### 3 Directed triangular random variable

Let us now review Van Assche's result for directed triangular random variables.

**Theorem 3.1.** If  $X_1$  and  $X_2$  are independent random variables with a common distribution  $F_X$ , then the characterizations of  $Z_1$  for  $n = 1$  and  $n = 2$  are identical.

**Proof.** We note that  $X_1$  and  $X_2$  have a common distribution function  $F_X$ . By using Lemma 2.2 for  $n = 2$ , we have

$$-\frac{1}{2} \mathcal{S}''(F_{Z_1}, z) = \mathcal{S}(F_X, z) \mathcal{S}'(F_X, z),$$

and so

$$-\mathcal{S}''(F_{Z_1}, z) = \frac{d}{dz} \mathcal{S}^2(F_X, z),$$

and

$$-\mathcal{S}'(F_{Z_1}, z) = \mathcal{S}^2(F_X, z). \quad (3.1)$$

We note that the Stieltjes transform tends to zero, when  $z$  is sufficiently large. In that case the constant in the differential equation will be zero. The equation (3.1) is exactly the equation obtained by Van Assche (1987) when  $X_1$  and  $X_2$  have a common distribution; so his results hold in our framework as well.  $\square$

This clever proof is due to the anonymous referee. Now, we apply Lemma 2.2 for some characterizations, when  $X_1$  and  $X_2$  are not identically distributed.

**Theorem 3.2.** Let  $X_1$  and  $X_2$  be independent random variables and  $Z_1$  be a directed triangular random variable satisfying (1.1). For  $n = 2$ , we have,

(a) if  $X_1$  has uniform distribution on  $[-1, 1]$ , then  $Z_1$  has semicircle distribution on  $[-1, 1]$  if and only if  $X_2$  has Arcsin distribution on  $[-1, 1]$ ;

(b) if  $X_1$  has uniform distribution on  $[-1, 1]$ , then  $Z_1$  has power semicircle distribution if and only if  $X_2$  has power semicircle distribution, i.e.,

$$f(z) = \frac{3(1 - z^2)}{4}, \quad -1 \leq z \leq 1;$$

(c) if  $X_1$  has Beta(1, 1) distribution on  $[0, 1]$ , then  $Z_1$  has Beta( $\frac{3}{2}$ ,  $\frac{3}{2}$ ) distribution if and only if  $X_2$  has Beta( $\frac{1}{2}$ ,  $\frac{1}{2}$ ) distribution;

(d) if  $X_1$  has uniform distribution on  $[0, 1]$ , then  $Z_1$  has Beta(2, 2) distribution if and only if  $X_2$  has Beta(2, 2) distribution.

**Proof.** (a) For the “if” part we note that the random variable  $X_1$  has uniform distribution and  $X_2$  has arcsin distribution on  $[-1, 1]$ ; then

$$\mathcal{S}(F_{X_1}, z) = \frac{1}{2}(\ln |z + 1| - \ln |z - 1|),$$

and

$$\mathcal{S}(F_{X_2}, z) = \frac{1}{\sqrt{z^2 - 1}}.$$

From Lemma 2.2 and substituting the corresponding Stieltjes transforms of distributions, we get

$$\mathcal{S}''(F_{Z_1}, z) = \frac{2}{(z^2 - 1)^{\frac{3}{2}}}.$$

The solution  $\mathcal{S}(F_{Z_1}, z)$  is

$$\mathcal{S}(F_{Z_1}, z) = 2(z - \sqrt{z^2 - 1}),$$

which is the Stieltjes transform of the semicircle distribution on  $[-1, 1]$ .

For the “only if” part we assume that the random variable  $Z_1$  has semicircle distribution. Then it follows from lemma 2.2 that

$$\mathcal{S}(F_{X_2}, z) \frac{1}{1 - z^2} = \frac{-1}{(z^2 - 1)^{\frac{3}{2}}}.$$

The proof is completed.

(b) By an argument similar to that given in (a) and solving the following differential equations,

$$S''(F_Z, z) = \frac{2}{(z^2 - 1)} \left( \frac{3z}{2} + \frac{3}{4}(1 - z^2)(\ln|z + 1| - \ln|z - 1|) \right), \text{ (for the “if” part), and}$$

$$\frac{1}{1 - z^2} S(F_{X_2}, z) = \frac{3}{4} \frac{2z + (1 - z^2)(\ln|z + 1| - \ln|z - 1|)}{(1 - z^2)}, \text{ (for the “only if” part),}$$

the proof can be completed.

(c) By Lemma (2.2), we have

$$-\frac{1}{2} S''(F_Z, z) = \frac{-1}{z(z - 1)} \frac{1}{\sqrt{z(z - 1)}}, \text{ (for the “if” part), and}$$

$$\frac{-1}{z(z - 1)} S(F_{X_2}, z) = \frac{-1}{z(z - 1) \sqrt{z(z - 1)}}, \text{ (for the “only if” part).}$$

The proof can be completed by solving the above differential equations.

(d) By Lemma (2.2), we have

$$S''(F_{Z_1}, z) = \frac{-2}{z(z - 1)} (6(z^2 - z)(\ln|z| - \ln|z - 1|) - 6z + 3), \text{ (for the “if” part), and}$$

$$\mathcal{S}(F_{X_2}, z) = 6(z - z^2)(\ln|z| - \ln|z - 1|) + 6z - 3, \text{ (for the “only if” part).}$$

Solving the differential equations, can complete the proof. □



## 4 TSP random variables

In section 3, we used a powerful method, based on the use of Stieltjes transforms, to obtain the distribution of  $Z_1$  given in (1.1). It seems that one can not use that method to find the distribution of  $Z_2$  given in (1.2). So we employ a direct method to find the distribution of  $Z_2$ .

Let us follow Lemma 4.1 to find a simple method to get the distribution of  $Z_2$ . The work of Soltani and Homei (2009b) leads us to the following lemma.

**Lemma 4.1.** Suppose  $W$  has a power distribution with parameter  $n$ ,  $n \geq 1$ ,  $n$  is an integer, and let  $Y_1 = \text{Min}(X_1, X_2)$ ,  $Y_2 = \text{Max}(X_1, X_2)$ , where  $X_1$  and  $X_2$  are independent random variables. Let

$$X = Y_1 + W(Y_2 - Y_1). \quad (4.1)$$

Then

- (a)  $X$  is a TSP random variable.
- (b)  $X$  can be equivalently defined by

$$X = \frac{1}{2}(X_1 + X_2) + (W - \frac{1}{2})|X_1 - X_2|.$$

**Proof.** (a)

$$\begin{aligned} F_{X|X_1, X_2}(z) &= P(Y_1 + W(Y_2 - Y_1) \leq z | X_1 = x_1, X_2 = x_2) \\ &= P(y_1 + W(y_2 - y_1) \leq z) \\ &= \left( \frac{z - y_1}{y_2 - y_1} \right)^n. \end{aligned}$$

(b) The proof can be completed by substituting  $\text{Min}(X_1, X_2)$  and  $\text{Max}(X_1, X_2)$  with  $Y_1$  and  $Y_2$  in (4.1). □

### 4.1 Moments of TSP random variables

The following theorem provides equivalent conditions for  $\mu'_k = EZ_2^k$ .

**Theorem 4.1.1.** Suppose that  $Z_2$  is a TSP random variable satisfying (1.2). If  $X_1$  and  $X_2$  are random variables and  $E|X_i|^k < \infty$ ,  $i = 1, 2$ , for all integers  $k$ , then

- (a)  $EZ_2^k = n \frac{\Gamma(k+1)}{\Gamma(k+n+1)} \sum_{i=0}^k \frac{\Gamma(k-i+n)}{\Gamma(k-i+1)} E(Y_1^i Y_2^{k-i});$   
(b)  $EZ_2^k = \sum_{i=0}^k \binom{k}{i} (\frac{1}{2})^{k-i} E(W - \frac{1}{2})^i E(X_1 + X_2)^{k-i} |X_1 - X_2|^i;$   
(c)  $EZ_2^k = \sum_{i=0}^k \binom{k}{i} \frac{n}{n+i} E(Y_1^{k-i} (Y_2 - Y_1)^i).$

**Proof.** (a) By using Lemma 4.1, we obtain that

$$\begin{aligned} EZ_2^k &= E\left(\sum_{i=0}^k \binom{k}{i} (1-W)^i Y_1^i W^{k-i} Y_2^{k-i}\right) \\ &= \sum_{i=0}^k \binom{k}{i} E(W^{k-i} (1-W)^i) E(Y_1^i Y_2^{k-i}) \\ &= n \frac{\Gamma(k+1)}{\Gamma(k+n+1)} \sum_{i=0}^k \frac{\Gamma(k-i+n)}{\Gamma(k-i+1)} EY_1^i Y_2^{k-i}. \end{aligned}$$

(b) This can be easily proved by Lemma 4.1(b).

(c) It straightforwardly follows from (4.1).  $\square$

Let us consider expectation and variance of  $Z_2$ . First, we suppose that  $EY_1 = \mu_1$ ,  $EY_2 = \mu_2$ ,  $\text{Var}Y_1 = \sigma_1^2$ ,  $\text{Var}Y_2 = \sigma_2^2$ , and  $\text{Cov}(Y_1, Y_2) = \sigma_{12}$ . Then

$$EZ_2 = \frac{\mu_1 + n\mu_2}{n+1},$$

and also, if  $EX_1 = EX_2 = 0$ , then

$$E(Z_2) = EY_1 + \frac{n}{n+1}(EY_2 - EY_1).$$

By  $Y_1 + Y_2 = X_1 + X_2$ , we have

$$E(Z_2) = E(Y_1) + \frac{n}{n+1}(-2EY_1) = \frac{1-n}{1+n}EY_1. \quad (4.2)$$

It can easily follow from (4.2) that the Arcsin result of Van Assche (1987) is only true for  $n = 1$ , and also, one can see that Theorem (3.2) in section 3 does not hold for the above  $Z_2$ .

About the variance, we have

$$\text{Var}Z_2 = \frac{n(\mu_1 - \mu_2)^2 + n(n+1)^2\sigma_2^2 + 2(n+1)(\sigma_1^2 + n\sigma_{12})}{(n+1)^2(n+2)}.$$

Following the computation of expectation and variance, we evaluate them for some well-known distributions. If  $X_1$  and  $X_2$  have standard normal distributions, then from Theorem 4.1.19b) and the fact that  $X_1 - X_2$  and  $X_1 + X_2$  are independent, it follows that their first, second and third order moments are equal, respectively, to

$$\begin{aligned} EZ_2 &= \frac{1}{\sqrt{\pi}} \left( \frac{n-1}{n+1} \right), \\ EZ_2^2 &= \frac{n^2 + n + 2}{(n+1)(n+2)}, \text{ and} \\ EZ_2^3 &= \frac{1}{2\sqrt{\pi}} \frac{5n^3 + 12n^2 + 13n - 30}{(n+3)(n+2)(n+1)}. \end{aligned}$$

Also, in case  $X_1$  and  $X_2$  have uniform distributions, Theorem 4.1.1(b) implies that,

$$\begin{aligned} EZ_2^k &= n \frac{\Gamma(k+1)}{\Gamma(n+k+1)} \sum_{i=0}^k \frac{\Gamma(k-i+n)}{\Gamma(k-i+1)} \frac{2}{(k+2)(i+1)}, \\ EZ_2 &= \frac{2n+1}{3(n+1)}, \text{ and} \\ Var(Z_2) &= \frac{1}{18} \frac{n^3 + 3n^2 + 6n + 2}{(n+1)^2(n+2)}. \end{aligned}$$

Since some distributions do not have any moments, Theorem 4.1.1 is not applicable for investigating Van Assche's results for them, whence, we prove the following theorem:

**Theorem 4.1.2.** Suppose that  $Z_2$  is a TSP random variable satisfying (4.1). Then

- (a)  $Z_2$  is location invariant;
- (b) if  $X_1$  and  $X_2$  have symmetric distribution around  $\mu$ , then  $Z_2$  has symmetric distribution around  $\mu$ , only when  $n = 1$ .

**Proof.**

- (a) Is immediate.

(b) We can assume without loss of generality that  $\mu = 0$ . If  $Z_2$  has a symmetric distribution around zero, then

$$Y_1 + W(Y_2 - Y_1) \stackrel{d}{=} -[Y_1 + W(Y_2 - Y_1)].$$

We note that

$$Y_1 + W(Y_2 - Y_1) \stackrel{d}{=} [-Y_1 + W(-Y_2 - (-Y_1))].$$

Since,  $-\text{Min}(X_1, X_2) = \text{Max}(-X_1, -X_2)$ ,  $X_1 \stackrel{d}{=} -X_1$  and  $X_2 \stackrel{d}{=} -X_2$ , we have

$$Y_1 + W(Y_2 - Y_1) \stackrel{d}{=} Y_2 + W(Y_1 - Y_2). \quad (4.3)$$

By equating the conditional distributions given at  $X_1 = x_1$  and  $X_2 = x_2$  in (4.2), we conclude that  $n = 1$ .  $\square$

It can also easily follow from Theorem (4.1.1) that the Cauchy result of Van Assche (1987) is true only for  $n = 1$ .

## 4.2 Distributions of TSP random variables

In this subsection, we investigate computing distributions by the direct method. We will give two examples of derivation based on (4.1). This method may be complicated in some cases, but we have chosen some easy to follow examples.

**Example 4.2.1.** Let  $X_1, X_2$  and  $W$  be independent random variables such that  $X_1$  and  $X_2$  are uniformly distributed over  $[0, 1]$ , and  $W$  has a power function distribution with parameter  $n$ . We find the value  $f_{Z_2}(z; n)$  by means of  $f_{Z_2|W}(z|w)$ ; therefore

$$f_{Z_2|W}(z|w) = \begin{cases} \frac{2z}{w}, & 0 < z < w, \\ \frac{2(1-z)}{1-w}, & w < z < 1. \end{cases} \quad (4.4)$$

By using the distribution of  $W$ , the density function  $f_{Z_2}(z; n)$  can be expressed in terms of the Gauss hypergeometric function  $F(a, b, c; z)$ , which is a well-known special function. Indeed according to Euler's formula, the Gauss hypergeometric function assumes the integral representation

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt,$$

where  $a, b, c$  are parameters subject to  $-\infty < a < +\infty$ ,  $c > b > 0$ , whenever they are real, and  $z$  is the variable (see Zayed 1996). By using Euler's formula, the density function of  $Z_2$  can be expressed as follows:

$$f_{Z_2}(z; n) = \frac{2nz}{n-1} (1 - z^{n-1}) + 2(1-z)z^n F(1, n, n+1, z), \quad 0 < z < 1, \quad (4.5)$$

where  $n > 0$  and  $n \neq 1$ . When  $n = 1$ , similar calculations lead to the following distribution

$$f_{Z_2}(z) = -2(1-z)\log(1-z) - 2z\log(z), \quad 0 < z < 1.$$

The probability density function  $f_{Z_2}(z)$  was introduced by Johnson and Kotz (1990), for the first time, under the title “uniformly randomly modified time”. So  $f_{Z_2}(z; n)$  can be seen as an extension of the above mentioned distribution. We note that, from (4.1) and a simple Monte Carlo procedure using only simulated uniform variables, one is able to simulate the distribution (4.5).

**Example 4.3.1.** Let  $X_1$  and  $X_2$  be independent random variables with Beta(1, 2) distribution. Then if  $W$  has Beta(3, 1) distribution,  $Z_1$  has Beta(2, 3) distribution.

In the following theorem we compute the Stieltjes transform of  $Z_2$  for  $n = 2$ . Let us remark that the complexity of the integral in the theorem indicates that for this case the direct method is preferred.

**Theorem 4.4.1** Let  $Z_2$  be a undirected triangular random variable that satisfies (1.2). Suppose that the random variables  $X_1$  and  $X_2$  are independent and continuous with the distribution functions  $F_{X_1}$  and  $F_{X_2}$ , respectively. Then

$$-\frac{1}{2}\mathcal{S}'''(F_Z, z) = \mathcal{S}'(F_{X_1}, z)\mathcal{S}'(F_{X_2}, z) + 2\mathcal{S}(F_{X_1}, F_{X_2}, z),$$

where

$$\mathcal{S}(F_{X_1}, F_{X_2}, z) = \int_{\mathbb{R}^2} \frac{1}{(z-x_1)(z-x_2)(x_2-x_1)^2} \prod_{i=1}^2 F_{X_i}(dx_i).$$

**Proof.** By using an argument similar to that given in Section 3, we can conclude that

$$\int f(x) dF_{Z_2}^{(2)}(x) = \int_{\mathbb{R}^2} F_{Z_2|x_1, x_2}(f) \prod_{i=1}^2 F_{X_i}(dx_i).$$

So,

$$-\frac{1}{2}\mathcal{S}'''(F_{Z_2}, z) = \int_{\mathbb{R}^2} F_{Z_2|x_1, x_2}(g_z) \prod_{i=1}^2 F_{X_i}(dx_i),$$

for  $g_z(x) = \frac{1}{(z-x)^2}$ . From

$$F_{Z_2|x_1, x_2}(g_z) = \frac{\frac{1}{(z-x_1)^2}}{(x_2-x_1)^2} + \frac{\frac{1}{(z-x_2)^2}}{(x_1-x_2)^2}$$

and by using partial fractional rule, we have

$$F_{Z_2|x_1,x_2}(g_z) = \frac{1}{(z-x_1)^2(z-x_2)^2} + \frac{2}{(x_2-x_1)^2} \frac{1}{(z-x_1)(z-x_2)}.$$

Therefore,

$$-\frac{1}{2}\mathcal{S}'''(F_{Z_2}, z) = \int_{\mathbb{R}^2} \left( \frac{1}{(z-x_1)^2(z-x_2)^2} + \frac{2}{(x_2-x_1)^2(z-x_1)(z-x_2)} \right) \prod_{i=1}^2 F_{X_i}(dx_i),$$

and

$$-\frac{1}{2}\mathcal{S}'''(F_{Z_2}, z) = \mathcal{S}'(F_{X_1}, z)\mathcal{S}'(F_{X_2}, z) + 2\mathcal{S}(F_{X_1}, F_{X_2}, z).$$

This finishes the proof.  $\square$

It is worth mentioning that the present method yields other extensions too; the following is such an example.

**Example 4.3.2.** Suppose that  $X_1, X_2, W$  are independent random variables. If  $X_1$  and  $X_2$  have uniform distributions on  $[0, 1]$  and  $W$  has Beta(2, 2) distribution, then  $Z_2$  has the same distribution as  $W$ .

If the product moments of order statistics are known, those of  $W$  can be derived from that of  $Z_2$  by using Theorem 4.1.1(a). Then the distribution of  $W$  is characterized by that of  $Z_2$ .

By an argument similar to the one given in Example 4.2.1, when  $W$  has a Beta distribution with parameters  $n$  and  $m$ , we find the distribution  $f_{Z_2}(z; n, m)$  as

$$\frac{B(n-1, m)}{B(n, m)} 2z(1 - I_z(n-1, m)) + \frac{B(n, m-1)}{B(n, m)} 2(1-z)I_z(n, m-1), \quad 0 < z < 1,$$

where  $I_x(a, b)$  is incomplete Beta function:

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad (a, b > 0).$$

## 5 Conclusions

We have described how (a) methods of Stieltjes transform, and (b) directed methods, could be used for obtaining the distributions, characterizations and properties of the random mixture of variables

defined in (1.1) and (1.2). Of course each one of the methods (a) or (b) has its own advantages and disadvantages, and none of them has a preference over the other. The TSP random variable when  $X_1$  and  $X_2$  have uniform distributions, led us to a new family of distributions which can be regarded as some generalization of “uniformly randomly modified time”. The proposed model in the direct method can easily lead to distribution generalizations, though this is not possible for the first method, but here the characteristics can be easily computed.

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